SUMMARY  This paper analyzes the impact of directional antennas in improving the transmission capacity, defined as the maximum allowable spatial node density of successful transmissions multiplied by their data rate with a given outage constraint, in wireless networks. We consider the case where the gain $G_a$ for the mainlobe of beamwidth can scale at an arbitrarily large rate. Under the beamwidth scaling model, the transmission capacity is analyzed for all path-loss attenuation regimes for the following two network configurations. In dense networks, in which the spatial node density increases with the antenna gain $G_a$, the transmission capacity scales as $G_a^{\alpha}$, where $\alpha$ denotes the path-loss exponent. On the other hand, in extended networks of fixed node density, the transmission capacity scales logarithmically in $G_a$. For comparison, we also show an ideal antenna model where there is no sidelobe beam. In addition, computer simulations are performed, which show trends consistent with our analytical behaviors. Our analysis sheds light on a new understanding of the fundamental limit of outage-constrained ad hoc networks operating in the directional mode.

key words: beamwidth scaling, directional antenna, outage constraint, scaling law, transmission capacity, wireless network

1. Introduction

The transmission capacity characterizes the area spectral efficiency, defined as the maximum allowable spatial node density of successful transmissions multiplied by their data rate subject to an outage constraint in wireless ad hoc networks [1]. In [2], the transmission capacity is collectively analyzed in a variety of network scenarios by using various techniques such as spread spectrum [1], [3], spectrum allocation (e.g., spectrum sharing [4] and spectrum partitioning [5], [6]), successive interference cancellation [7], joint detection [8], power control [9], threshold scheduling [10], and multiple antennas [11], [12]. Complementary to the transmission capacity research, an alternative approach to characterizing the performance of ad hoc networks has been introduced by analyzing the capacity scaling law, so-called the transport capacity, that quantifies how the aggregate capacity grows with the number of nodes [13]. The transport capacity scaling in a network of unit area was shown to be achieved using the nearest-neighbor multihop routing scheme (also known as Gupta–Kumar routing scheme). Multihop schemes were further studied and analyzed in [14]–[18]. There has also been research on improving the transport capacity up to a linear scaling in the number of nodes by using novel techniques such as networks with node mobility [19], interference alignment [20], directional antennas [21], [22], and infrastructure support [23], [24]. Within the transport capacity framework [13]–[24], idealized centralized scheduling is used, where it can eliminate outages by carefully determining the optimal set of transmitters in each time slot, which is not feasible in practice. On the other hand, the transmission capacity framework in [1]–[12] allows the use of only uncoordinated transmissions, which fails to schedule simultaneous transmissions with the objective of controlling interference levels, thus causing outages.

1.1 Related Work

In the literature, studies on the transmission capacity have focused mostly on wireless network scenarios using omnidirectional antennas at each node. However, for wireless systems using millimeter wave (mmWave) technologies [25] operating in the 10–300 GHz band, which have been considered as one solution enabling gigabit-per-second data rates, equipping directional antennas at each node may be more challenging. This is because mmWave links are inherently directional and thus steerable antenna arrays can be easily implemented, thereby resulting in a much higher link gain [26]. For this reason, the use of directional antennas in ad hoc networks has recently emerged as a promising technology leading to higher spatial reuse ratios, improved transmission distances, and reduced interference levels at low cost [21], [22], [27]–[33]. Specifically, the previous work on the transport capacity scaling based on the directional protocol model [27] showed that, for an infinitely large antenna gain (or equivalently, for an infinitely small beamwidth), directional antennas offer magnificent potential for improving the throughput performance up to a linear scaling in the number of nodes, which is the best we can hope for. Similarly, the transport capacity was studied based on a general interference model for directional antennas [21], [22], [28]. Other studies have examined the transport capacity when directional antennas are used under different assumptions as well as different situations [29]–[31]. In addition, in [32], the transport capacity based on the information-theoretic approach was recently analyzed by introducing the elastic multihop routing protocol in wireless networks equipping directional antennas. In [33], in the directional mode, the effects of beam misdirection on the transmission capaci-
ity were investigated, and capacity-maximizing beamwidths were found in the presence of orientation error.

1.2 Main Contributions

In this paper, we analyze in detail, for the first time, how the transmission capacity scaling behaves in large-scale outage-constrained wireless networks when a single directional antenna is equipped at each node. We use a simplified hybrid antenna model with both mainlobe and sidelobe, where the gain $G_m$ for the mainlobe of beamwidth is assumed to scale at an arbitrarily large rate as the beamwidth becomes narrower. The impact of such a beamwidth scaling in improving the transmission capacity subject to an outage constraint is analyzed in both dense and extended network configurations. For all path-loss attenuation regimes, we first derive upper and lower bounds on the outage probability by characterizing two different types of interference. Based on closed-form expressions of the two bounds on the outage probability, it is shown that the transmission capacity scales as $G_m^{4/\alpha}$ in dense networks where the spatial node density increases with the antenna gain $G_m$, whereas, in extended networks of fixed node density where the network size is proportional to the number of nodes, the transmission capacity scales as $\log G_m$. These results indicate that using directional antennas can provide a significantly high node density (i.e., a lot of simultaneous transmission pairs) in dense networks and an increased data rate of each transmission pair up to a logarithmic scale in extended networks. Thus, if such a steerable directional antenna system with a narrow beamwidth is implementable, then a substantial network throughput gain can be attainable even with no use of multiple antennas. For comparison, we also analyze an ideal antenna model where there is no sidelobe beam. Numerical results are provided for both dense and extended networks to validate the transmission capacity scaling in practical antenna gain regimes.

Throughout this paper, $\mathbb{P}(-)$ and $\mathbb{E}[-]$ denote the probability and statistical expectation, respectively. Unless otherwise stated, all logarithms are assumed to be to the base 2.

1.3 Organization

The rest of this paper is organized as follows. In Sect. 2, we introduce the system model. In Sect. 3, the transmission capacity scaling laws are analyzed for both dense and extended networks. An ideal scenario with no sidelobe is analyzed and numerical evaluation is shown in Sect. 4. Finally, we summarize the paper with some concluding remarks in Sect. 5.

2. System Model

As in [1], [2], we consider an ad hoc wireless network, where the distribution of nodes in the network follows a homogeneous Poisson point process (PPP) of density $\lambda$ on a plane $\mathbb{R}^2$. We pick pairings so that each transmitter is assigned to one receiver at a fixed distance $r > 0$ (refer to Fig. 1).

Each node is equipped with a single directional antenna and transmits with constant power $P$.

As illustrated in Fig. 2, we define a hybrid antenna model whose mainlobe is characterized as a sector and whose sidelobe forms a circle (backlobes are ignored in this model). The antenna beam pattern has a gain value $G_m$ for the mainlobe of beamwidth $\theta \in (0, 2\pi)^{[1]}$, and also has a sidelobe of gain $G_s$, of beamwidth $2\pi - \theta$. The parameters $G_m$ and $G_s$ are then related according to

$$\frac{\theta}{2\pi} G_m + \frac{2\pi - \theta}{2\pi} = \frac{1}{G_s},$$

where $0 < G_s \leq 1 \leq G_m$. In our work, we assume that $G_m = \Theta(1/\theta)$ and $G_s = \Theta(1)^{[11]}$, which does not violate the law of energy conservation. For simplicity, we assume unit antenna efficiency, i.e., no antenna loss. Nodes can use their antennas for directional transmission or directional reception, where the transmitter and receiver antenna gains are assumed to be the same. We also assume that each antenna is steerable, i.e., each node can point its antenna in any desired direction. Packets are delivered to the corresponding receiver (destination) using a single-hop transmission.

Instead of directional antennas modeled as a cone in a three-dimensional view, we use a rather simple two-dimensional antenna model since simplifying the shape of the antenna pattern will not cause any fundamental change in term of scaling law.

Our system model can be extended to multi-antenna scenarios, but such an extension does not change our scaling results if the number of antennas at each node is independent of $G_m$.

We use the following notation: i) $f(x) = O(g(x))$ means that there exist constants $C$ and $c$ such that $f(x) \leq C g(x)$ for all $x > c$, ii) $f(x) = \Omega(g(x))$ means that $f(x) \geq C g(x)$, and iii) $f(x) = \Theta(g(x))$ if $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$ [34].
which may be feasible with the help of directional antennas.

Let \( p = \frac{\theta}{2\pi} \) denote the probability that a transmitter is located in the coverage of the mainlobe of an arbitrary receiver. Let \( \Pi_1, \Pi_2, \) and \( \Pi_3 \) denote three different sets of simultaneously transmitting nodes according to beam directions—specifically, it follows that both nodes, \( i_1 \in \Pi_1 \) and the reference receiver, beamform to each other with probability \( p^2 \), either \( i_2 \in \Pi_2 \) or the reference receiver beamforms to the other node (but not both) with probability \( 2p(1-p) \), and neither \( i_3 \in \Pi_3 \) nor the reference receiver beamforms to the other node with probability \( (1-p)^2 \), as illustrated in Fig. 3.

Now, let us turn to channel modeling. We consider only path-loss attenuation effects, and ignore additional channel effects such as small-scale fading, which may be negligible as long as scaling law is concerned. Then, the received signal power from the intended transmitter is

\[
P_r = PG_m^2r_i^{-\alpha}, \tag{1}
\]

and the total amount of interference at the reference receiver from all simultaneously transmitting nodes is

\[
I = \sum_{i_1 \in \Pi_1} PG_m^2r_i^{-\alpha} + \sum_{i_2 \in \Pi_2} PG_m^2G_sG_r^{-\alpha} + \sum_{i_3 \in \Pi_3} PG_m^2r_i^{-\alpha}, \tag{2}
\]

where the parameter \( r_i \) \((k = 1, 2, 3)\) denotes the distance between transmitter \( i \) in the set \( \Pi_k \) and the reference receiver, and \( \alpha > 2 \) denotes the path-loss exponent. We assume that the additive ambient noise power is denoted by \( \eta \). We do not assume the use of any multiuser detection schemes at each receiver while treating interference as noise.

### 3. Transmission Capacity Scaling

In this section, we present the transmission capacity scaling law as a function of the gain \( G_m \) for two different types of wireless networks (i.e., dense and extended networks) using directional antennas. As in [1]–[4], [7], [9], [11], we define the transmission capacity as

\[
C_T = R(1 - \epsilon) \text{ [bits/sec/Hz/m}^2\text{]},
\]

where \( R = \log_2(1 + \beta) \) is the common data rate per pair. Here, \( \epsilon \) and \( \beta \) denote the outage probability and a target threshold of the received signal-to-interference-and-noise-ratio (SINR) required for successful reception, respectively. An attempted transmission is successful if the received SINR is above \( \beta \); otherwise, the transmission fails, i.e., an outage occurs.

#### 3.1 Outage Analysis

Let us first focus on computing outage probabilities, which are essential to find the transmission capacity. The target outage probability is defined as

\[
\epsilon = \mathbb{P}\left(\frac{P_r}{I + \eta} < \beta\right), \tag{3}
\]

where \( P_r \) and \( I \) are given in (1) and (2), respectively. Using (1), (3) can be rewritten as

\[
\epsilon = \mathbb{P}(\gamma > T_d), \tag{4}
\]

where \( \gamma = \frac{P_r}{\eta} \) and

\[
T_d = \frac{G_m^2r_i^{-\alpha}}{\beta} - \frac{\eta}{P_r}. \tag{5}
\]

In the omnidirectional mode, closed-form expressions of the outage probability were analyzed only for the path-loss exponent \( \alpha = 4 \) [2]. It is, however, analytically intractable to obtain the exact outage probability for general \( \alpha \) values. Instead, in our work, similarly as in [1], [3], [4], [7], [9], [11], both upper and lower bounds on the outage probability are characterized in the network model under consideration.

Now, we derive upper and lower bounds on the outage probability in the directional mode. Under our large-scale fading model, outage can be determined depending heavily on the positions of transmitting nodes (i.e., interferers) located according to a PPP on \( \mathbb{R}^2 \). To give a lower bound on the outage probability, we first define the dominant interferer. An interferer is called dominant if its interference level, normalized by \( P_r \), that affects the reference receiver is greater than a threshold \( T_d \) in (5). Thus, the lower bound indicates how often there is at least one interferer that is strong enough to cause an outage, which is established in the following lemma. For notational convenience, we start from letting

\[
f(G_m) = pG_m^{-\frac{\alpha}{2}} + 2p(1-p)(G_mG_s)^{\frac{\alpha}{2}} + (1-p)^2G_s^{-\frac{\alpha}{2}}, \tag{6}
\]

where \( p = \frac{\theta}{2\pi} \).

**Lemma 1.** In the directional mode, a lower bound on the outage probability is given by

\[
\epsilon_l = 1 - \exp\left(-\frac{\lambda \pi}{T_d^{2\alpha}/f(G_m)}\right), \tag{7}
\]

where \( T_d \) and \( f(G_m) \) are given in (5) and (6), respectively.

**Proof.** Let \( h_k \) and \( r_k \) denote the channel gain and distance between transmitter \( i \in \Pi_k \) \((k = 1, 2, 3)\) and the reference receiver, respectively. Then, it follows that

\[
h_k = \begin{cases} G_m^2 & \text{if } k = 1 \\ G_m^2G_s & \text{if } k = 2 \\ G_s^2 & \text{if } k = 3. \end{cases} \tag{8}
\]

From (2), (4), and (8), the probability that an outage is caused by a certain one dominant interferer \( i \in \Pi_k \) is given by

\[
\epsilon = \mathbb{P}(h_k r_i^{-\alpha} > T_d) 
= \mathbb{P}(r_i < d_k), \tag{9}
\]

where
The outage probability is given by

\[ P_{\text{out}} = \mathbb{P}(\gamma < \gamma_{\text{th}}) = \mathbb{P}(\gamma < \gamma_{\text{th}}|\Pi_{\text{dom}} = \emptyset) \mathbb{P}(\Pi_{\text{dom}} = \emptyset) + \mathbb{P}(\gamma < \gamma_{\text{th}}|\Pi_{\text{dom}} \neq \emptyset) \mathbb{P}(\Pi_{\text{dom}} \neq \emptyset) \]

where \( \mathbb{P}(\Pi_{\text{dom}} = \emptyset) = \exp(-\lambda \pi d_k^2) \) (refer to [1] for the detailed proof). Thus, the probability that there is no dominant interferer over the whole network is given by

\[ P(\Pi_{\text{dom}} = \emptyset) = \mathbb{P} \left(\bigcap_{k=1}^{K} \Pi_k \right) = \prod_{k=1}^{K} \mathbb{P}(\Pi_k = \emptyset) = \prod_{k=1}^{K} \exp(-\lambda \pi d_k^2) \]

Using (11) finally leads to (7), which completes the proof of the theorem. \( \square \)

Next, we derive an upper bound on the outage probability by considering both dominant and non-dominant interferers. An interferer is called non-dominant if it is located outside the disk depicted in Fig. 3.

**Lemma 2.** In the directional mode, an upper bound on the outage probability is given by

\[ e_u = e_l + \frac{2\pi \lambda (1 - e_l)}{(\alpha - 2) T_d^{2/\alpha}}, \tag{11} \]

where \( T_d \) and \( e_l \) are given in (5) and (7), respectively.

**Proof.** Let \( \Pi_{\text{dom}} \) denote the set consisting of all dominant interferers deployed over the network. Then, it follows that

\[ P(\gamma < \gamma_{\text{th}}) = P(\gamma < \gamma_{\text{th}}|\Pi_{\text{dom}} = \emptyset)P(\Pi_{\text{dom}} = \emptyset) + P(\gamma < \gamma_{\text{th}}|\Pi_{\text{dom}} \neq \emptyset)P(\Pi_{\text{dom}} \neq \emptyset) \]

where \( e_l \) is the outage probability caused by the dominant interferers, as derived in (7), and \( \gamma_{\text{dom}} \) represents the aggregate interference (normalized by \( P \)) created by all non-dominant interferers. Here, the condition \( \Pi_{\text{dom}} = \emptyset \) indicates that there is no dominant interferer in the network, thereby yielding \( \gamma_{\text{dom}} = \gamma \) under the condition.

Now, let us turn to computing the term \( P(\gamma_{\text{dom}} > T_d) \), which is upper-bounded by

\[ P(\gamma_{\text{dom}} > T_d) \leq \frac{\mathbb{E}[\gamma_{\text{dom}}]}{T_d} \]

due to the Markov inequality. From the fact that the condition that transmitter \( i \in \Pi_k \) is a non-dominant interferer is given by \( h_k R_{ik}^{\alpha} \leq T_d \) (refer to (9)), we have

\[ \mathbb{E}[\gamma_{\text{dom}}] = \mathbb{E}\left[ \sum_{k=1}^{3} \sum_{i \in \Pi_{\text{dom}}} h_k R_{ik}^{\alpha} \right] = 2\pi \lambda \int_0^{T_d} U^{-\alpha} \, dl \]

where the second equality holds due to the Campbell-Mecke theorem [35] applied to our PPP network. Since \( P(\Pi_{\text{dom}} = \emptyset) = 1 - e_l \), using (13) in (12) leads to the upper bound in (11), which completes the proof of the lemma. \( \square \)

Using Lemmas 1 and 2, we are ready to analyze transmission capacity scaling laws for both dense and extended network scenarios as the antenna gain \( G_m \) increases.

### 3.2 Dense Network Scenario

In dense networks, we are interested in how to maximally scale the node density \( \lambda \) with increasing \( G_m \) (i.e., how many transmission pairs can be activated simultaneously) subject to given outage constraint \( e \) and data rate \( R \). The following theorem establishes our first main result in the directional mode of dense networks.

**Theorem 1.** In dense networks, where the spatial node density increases with \( G_m \), the transmission capacity scales as

\[ C_T = \Theta\left( G_m^{4/\alpha} \right), \]

where \( \alpha \) denotes the path-loss exponent.

**Proof.** Under the assumed antenna model (i.e., \( G_m = \Theta(1/\theta) \) and \( G_s = \Theta(1) \)), it follows that \( f(G_m) = \Theta(1) \) from (6). Let us focus on deriving an upper bound on the transmission capacity. Using (7) in Lemma 1, we have
\[ \frac{1}{T_d^{2/\alpha}} = \Theta(1). \]

From this relationship between \( \lambda \) and \( T_d \), an upper bound on the maximum node density, \( \lambda_u \), is given by

\[ \lambda_u = \Theta \left( T_d^{2/\alpha} \right) = \Theta \left( G_m^{4/\alpha} \right), \]

where the second equality holds from (5). Thus, given outage \( \epsilon \) and data rate \( R \), the transmission capacity is upper-bounded by

\[ C_T = O(R \lambda_u(1 - \epsilon)) = O \left( G_m^{4/\alpha} \right). \]

Now, let us turn to computing a lower bound on the transmission capacity using Lemma 2. Using (7), (11) in Lemma 2 can be rewritten as

\[ \left( 1 - \frac{2\lambda \pi}{(\alpha - 2) T_d^{2/\alpha}} \right) \frac{1}{1 - \epsilon} = \exp \left( \frac{\lambda \pi}{T_d^{2/\alpha}} f(G_m) \right). \]  

For \( \alpha > 2 \), from the fact that the left hand side (LHS) of (14) scales as \( \Theta(1) \), it follows that

\[ \exp \left( \frac{\lambda \pi}{T_d^{2/\alpha}} f(G_m) \right) = \Theta(1), \]

which results in \( \frac{1}{T_d^{2/\alpha}} = O(1) \) using \( f(G_m) = \Theta(1) \). Thus, a lower bound on the maximum allowable node density, \( \lambda_l \), is given by

\[ \lambda_l = \Theta \left( T_d^{2/\alpha} \right) = \Theta \left( G_m^{4/\alpha} \right). \]

Hence, given outage \( \epsilon \) and data rate \( R \), the transmission capacity is lower-bounded by \( C_T = \Omega \left( G_m^{4/\alpha} \right) \), which completes the proof of the theorem.

This scaling result indicates that \( \Theta \left( G_m^{4/\alpha} \right) \) transmission pairs can be active at the same time in dense networks while maintaining a given outage constraint. It is also examined that the transmission capacity \( C_T \) is reduced with increasing \( \alpha \). This is because, from Lemma 1, when \( \alpha \) increases, the radius \( d_k \) of the dominant interferers’ disk increases, which yields that the probability that a transmitter becomes a dominant interferer increases. To maintain given an outage probability, the number of active transmission pairs in the network needs to be reduced, thus resulting in a decrement of the node density \( \lambda \).

### 3.3 Extended Network Scenario

In extended networks, the transmission capacity scaling depends on the common data rate \( R \). The following theorem presents the relationship between the transmission capacity \( C_T \) and the antenna gain \( G_m \) in extended networks.

**Theorem 2.** In extended networks of fixed node density, the transmission capacity scales as

\[ C_T = \Theta(\log G_m). \]

**Proof.** From the fact that \( f(G_m) = \Theta(1) \), let us first derive an upper bound on the transmission capacity. Under the extended network model, using (7) in Lemma 1 leads to

\[ T_d^{2/\alpha} = \Theta(1), \]

which follows that \( \beta = \Theta(G_m^2) \) from (5). Thus, an upper bound on the data rate, \( R_u \), is given by

\[ R_u = \Theta \left( \log(1 + \beta) \right) = \Theta \left( \log(1 + G_m^2) \right) = \Theta(\log G_m). \]

Hence, given \( \epsilon \) and \( \lambda \), we obtain

\[ C_T = O(R_u(1 - \epsilon)) = O(\log G_m). \]

Next, we focus on computing a lower bound on the transmission capacity. By following the same argument as in the dense network setup (refer to Theorem 1), we can show that (15) also holds for extended networks under the assumption that \( \alpha > 2 \) and \( G_m \gg 1 \), which thereby results in \( T_d^{2/\alpha} = \Theta(1) \). From the relationship between \( \beta \) and \( T_d \) in (5), the maximum allowable \( \beta \) is given by \( \beta = \Theta(G_m^2) \), which reveals that a lower bound on the data rate, \( R_l \), is

\[ R_l = \Theta(\log(1 + \beta)) = \Theta(\log G_m). \]

Hence, given \( \epsilon \) and \( \lambda \), it follows that \( C_T = \Omega(\log G_m) \). This completes the proof of the theorem.

From Theorem 2, one can show that the data rate \( R \) at each transmission pair can be boosted up to a logarithmic scale in \( G_m \) when the outage constraint is satisfied. In extended networks, it is seen that the transmission capacity scaling \( T_C \) does not depend on \( \alpha \). We also note that this result is the same as the case where there is no interference. Thus, no performance loss occurs compared with an ideal interference-free network as long as scaling law is concerned.

### 4. Discussion

In this section, we first analyze an ideal directional antenna scenario with no sidelobe gain. Then, we perform numerical evaluation to validate the analytical results in Sect. 3.

#### 4.1 No Sidelobe Model

For comparison, let us consider an ideal antenna model where there is no sidelobe gain \( (G_s = 0) \). Then, the mainlobe gain \( G_m \) is simply given by \( G_m = \frac{\pi}{\alpha} \). Under the model, the transmission capacity scaling laws are analyzed for both dense and extended network configurations, which are presented in the following two theorems.

**Theorem 3.** Suppose that \( G_s = 0 \). Then, in dense networks, the transmission capacity scales as \( C_T = O \left( G_m^2 \right) \) and \( C = \Omega \left( G_m^{4/\alpha} \right) \), where \( \alpha \) is the path-loss exponent.

**Proof.** Under the ideal antenna model (i.e., \( G_m = \Theta(1/\theta) \)
and $G_s = 0$), it follows that $f(G_m) = \Theta(G_m^{4/\alpha - 2})$ from (6).

We first derive an upper bound on the transmission capacity. Similarly as in the proof of Theorem 1, using (7) in Lemma 1, we have

$$\frac{\lambda G_m^{4/\alpha - 2}}{T_d^{2/\alpha}} = \Theta(1),$$

which yields

$$\lambda_u = \Theta\left(\frac{T_d^{2/\alpha}}{G_m^{4/\alpha - 2}}\right) = \Theta(G_m^2).$$

Thus, given $\epsilon_l$ and $R$, the transmission capacity is upper-bounded by $C_T = O(G_m^2)$.

Now, let us turn to computing a lower bound on the transmission capacity using Lemma 2. From (14), we can find the following two conditions:

$$\frac{1}{T_d^{2/\alpha}} = \Theta\left(\frac{\lambda}{G_m^{4/\alpha}}\right) = O(1), \quad (16a)$$

$$\frac{\lambda}{T_d^{2/\alpha}} f(G_m) = \Theta\left(\frac{\lambda}{G_m^{2/\alpha}}\right) = O(1), \quad (16b)$$

where the condition (16a) is obtained since the LHS of (14) scales as $\Theta(1)$ for $\alpha > 2$, and another condition (16b) holds due to (15). Here, the first equality in (16a) and (16b) comes from using (5) and $f(G_m) = \Theta(G_m^{4/\alpha - 2})$. Thus, a lower bound on the maximum allowable node density, $\lambda_l$, that satisfies both conditions in (16) for all values of $\alpha > 2$ is given by

$$\lambda_l = \Theta\left(G_m^{4/\alpha}\right).$$

Hence, given outage $\epsilon_u$ and data rate $R$, the transmission capacity is lower-bounded by $C_T = \Omega(G_m^{4/\alpha})$, which completes the proof of the theorem.

**Theorem 4.** Suppose that $G_s = 0$. Then, in extended networks, the transmission capacity scales as $C_T = \Theta(\log G_m)$.

**Proof.** We first focus on computing a upper bound on the transmission capacity. Using the fact that $f(G_m) = \Theta(G_m^{4/\alpha - 2})$ and (7) in Lemma 1, we have

$$T_d^{2/\alpha} = \Theta(G_m^{4/\alpha - 2}),$$

which results in $\beta = \Theta(G_m^\alpha)$. Then, an upper bound on the data rate, $R_u$, is given by

$$R_u = \Theta(\log(1 + G_m^\alpha)) = \Theta(\log G_m).$$

Hence, given $\epsilon_l$ and $\lambda$, we obtain $C_T = O(\log G_m)$.

A lower bound on the transmission capacity is also derived similarly as in Theorem 3. From (14), the following two conditions are found:

$$\frac{1}{T_d} = \Theta\left(\frac{\beta}{G_m^{4/\alpha}}\right) = O(1), \quad (17a)$$

where the first equality follows due to (5) and $f(G_m) = \Theta(G_m^{4/\alpha - 2})$. Thus, the maximum allowable $\beta$ satisfying (17a) and (17b) is lower-bounded by $\beta_l = \Theta(G_m^2)$, which then yields that a lower bound on the data rate, $R_l$, is $R_l = \Omega(\log G_m)$. Hence, given outage $\epsilon_u$ and intensity $\lambda$, the transmission capacity is lower-bounded by $C_T = \Omega(\log G_m)$, which completes the proof of the theorem.

We remark that, in dense networks, there is a gap between two bounds on the transmission capacity, whereas in extended networks, the result is identical to Theorem 2, which means that the existence of a sidelobe beam does not cause any performance loss in scaling law.

### 4.2 Numerical Evaluation

In this subsection, to validate our analytical results in practical operating regimes (i.e., finite $G_m$ regimes), we perform numerical evaluation via computer simulations. In our
Monte-Carlo simulation, the interference level in (2) is generated $10^3$ times by placing interfering nodes according to a PPP on $\mathbb{R}^2$. In dense networks, the target SINR threshold $\beta$ is assumed to be 10 dB, while in extended networks, the node density $\lambda$ is assumed to be $2.5 \times 10^{-5}$ nodes/m$^2$.

The transmission capacity is first evaluated with respect to the mainlobe gain $G_m$ in dense and extended networks. The system parameters are summarized in Table 1. In Fig. 4, when there is a nonzero sidelobe gain (i.e., $G_s \neq 0$), both upper and lower bounds on the transmission capacity are shown, respectively, using Lemmas 1 and 2. It is seen that the simulation result (the solid curve) lies between these two bounds. Moreover, the asymptotic result (the dashed curve with no mark) is obtained from Theorems 1 and 2 with a proper bias. Especially, Fig. 5(a) shows that, in dense networks, the actual transmission capacity is dramatically improved compared to the $G_s \neq 0$ case, due to the absence of sidelobe beam, and tends to scale as $G^2_m$.

Now, let us turn to numerically show how other system parameters affect the transmission capacity. Numerical evaluation of the transmission capacity is performed in dense and extended networks according to the path-loss exponent $\alpha$, the per-pair distance $r$, the outage probability $\epsilon$, and the sidelobe gain $G_s$, which is shown in Figs. 6–9, respectively, when $G_m = 25$ and the parameters in Table 1 are used except for a target parameter. Likewise, it is seen that for each case, the simulation result lies between the corresponding analytical upper and lower bounds.

In addition, numerical evaluation is performed under the channel model including shadowing effects. The random shadowing variable is assumed to have log-normal distribution, normalized to have mean one and parameterized by its standard deviation $\sigma$ expressed in dB scale (i.e., the logarithmic standard deviation), and to be independent across differ-
1. Numerical evaluation of the transmission capacity versus the per-pair distance $r$ for $G_m = 25$.

2. Numerical evaluation of the transmission capacity versus the outage probability $\epsilon$ for $G_m = 25$.

### Table 1  System parameters.

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<thead>
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<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>Per-pair distance</td>
<td>30 m</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>Outage constraint</td>
<td>0.05</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Power noise</td>
<td>$10^{-9}$ W/Hz</td>
</tr>
<tr>
<td>$G_s$</td>
<td>Sidelobe gain</td>
<td>0.1</td>
</tr>
<tr>
<td>$P$</td>
<td>Transmmit power</td>
<td>20 W</td>
</tr>
<tr>
<td>$W$</td>
<td>Bandwidth</td>
<td>1 GHz</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>path-loss exponent</td>
<td>4</td>
</tr>
</tbody>
</table>

It would also be worth to show analytical results (i.e., transmission capacity scaling laws) under the channel model with shadowing effects, which do not seem to be straightforwardly derived. We leave the above issue as our future work.

5. **Conclusion**

In this paper, we have presented the effects of the beamwidth scaling on the transmission capacity in wireless networks with directional antennas. First, the outage analysis for the omnidirectional mode was extended to the directional mode by defining a hybrid antenna model. Under the beamwidth scaling model, the transmission capacity was then analyzed for all path-loss attenuation regimes in both dense and extended networks. Specifically, it turned out that, in dense networks, the transmission capacity scales as $G_m^4/\alpha$, whereas, in extended networks, it scales as $\log G_m$. The transmission capacity was also derived for the case where there is no sidelobe gain. Specifically, under no sidelobe model, upper and lower bounds on the transmission capacity scaling were derived in dense networks, and the scaling result in extended networks was shown to be identical to the case where there exists a nonzero sidelobe gain at each node. Our analytical
results were comprehensively validated by computer simulations.

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References


DO and SHIN: BEAMWIDTH SCALING IN WIRELESS NETWORKS WITH OUTAGE CONSTRAINTS


